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On polynomial approximation of circular arcs and helices

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ABSTRACT

We present a simple method for polynomial approximation of circular arcs and helices by expressing the trigonometric functions using the two-point Taylor expansion. We obtain the degree-$(2n + 1)$ polynomial for the approximation problem in an efficient way, which is very convenient to increase the degree of polynomial by adding new terms. An upper bound on the approximation error is available, so that we can obtain the lowest degree polynomial curve that can approximate a circular arc or helix segment within any user-prescribed error tolerance.

1. Introduction

Circles and circular arcs are widely used in geometric modeling and in problems of shape preservation due to their constant curvature. A helix segment is a natural generalization of circular arc in 3D space with constant curvature and torsion. Helices have been widely used for the tool path description, the simulation of kinematic motion, the design of highways, etc.

A circular arc can be exactly represented by conics or rational quadratic curves, or can be approximated by polynomial curves of degrees 3, 4 and 5 [1–4] and by cubic *B*-spline curves [5]. Piegl and Tiller [6] specifically addressed the problem of how to obtain a good (parametrically continuous) circle approximation. Floater [7] presented high order geometric Hermite interpolation of conic sections. A helix segment cannot be exactly represented by polynomials and rational polynomials. The approximation of the helix with rational polynomial curves has been extensively studied in the literature [8–12]. Mick and Röschel [8] presented a direct approach to interpolate the helix by cubic rational curves. Juhász [9] studied the same problem and proposed two methods with error estimation. And Seemann [10] focused on the rational approximation of degrees 4, 5 and 6. High accuracy approximation by quintic curves was investigated by Yang [11]. Ahn [12] constructed a simple approximation using conic and quadratic Bézier curves. Although the rational polynomial approximation has consistently higher accuracy than the polynomial approximation, the rational representation suffers from several drawbacks in practical applications. Perhaps the most important one is the lack of a good parameterization [13].

We assume without loss of generality that the circular arc and the helix segment are defined by

$$\mathbf{c}(\alpha; t) = (r\cos t, r\sin t) \quad \text{and} \quad \mathbf{h}(\alpha; t) = (r\cos t, r\sin t, pt), \quad t \in [-\alpha, \alpha], \tag{1}$$

where α, r and p are positive real numbers. Such expressions can always be achieved via rotation and translation.

The main idea of this paper is to approximate the trigonometric functions in terms of the degree-$(2n + 1)$ Hermite interpolating polynomials using the two-point Taylor expansion proposed by López and Temme [14]. Different from the standard Hermite representation (see e.g. [15]), the two-point Taylor expansion has a nice property: if we have calculated the degree-$(2n + 1)$ Hermite interpolant, it will be very convenient to obtain the degree-$(2n + 3)$ Hermite interpolant by simply adding one more new term to it, without having to recompute the whole. We obtain the Hermite interpolating

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polynomial of degree $2n+1$ for a circular arc or helix segment in a unified form. The main contributions are summarized as follows:

- The approximating polynomials are expressed like the Taylor series. It is very convenient to deal with polynomials having different degrees and to take differentiation and integration operations.
- The approximation error between the polynomial and the circular arc decreases rapidly and converges to zero as the polynomial degree tends to ∞.
- It can be considered as an adaptive method. For any user-prescribed error tolerance, the lowest degree polynomial can be found so that it is able to approximate the circular arc or helix segment within the tolerance.

2. Polynomial approximation method

For the trigonometric functions, $\cos t$ and $\sin t$, $t \in [-\alpha, \alpha]$, we want to obtain the degree-$(2n+1)$ Hermite interpolating polynomials $X_{2n+1}(\alpha; t)$ and $Y_{2n+1}(\alpha; t)$, respectively. The polynomial $X_{2n+1}(\alpha; t)$ (or $Y_{2n+1}(\alpha; t)$) interpolates all the first n derivatives of the function $\cos t$ (or $\sin t$) at $t = -\alpha, \alpha$. It is a Hermite interpolation problem, so they are uniquely determined from the interpolation conditions. Obviously, when $n = 0$, the degree-1 Hermite interpolants are

$$X_1(\alpha; t) = \cos\alpha \quad \text{and} \quad Y_1(\alpha; t) = \frac{\sin\alpha}{\alpha} t. \tag{2}$$

According to Definition 1 in [14], the degree-$(2n+1)$ Hermite interpolants are expressed by

$$X_{2n+1}(\alpha; t) = \cos\alpha + 2\alpha \sum_{k=1}^{n} x_k \left(t^2 - \alpha^2\right)^k \tag{3}$$

and

$$Y_{2n+1}(\alpha; t) = \frac{\sin\alpha}{\alpha} t + 2t \sum_{k=1}^{n} y_k \left(t^2 - \alpha^2\right)^k, \tag{4}$$

where the coefficients x_k and y_k are given by

$$x_k = \sum_{i=0}^{k} \frac{(k+i-1)!}{i!(k-i)!k!} \frac{(-1)^{k+1} k \cos^{(k-i)}(\alpha) + (-1)^i i \cos^{(k-i)}(-\alpha)}{(-2\alpha)^{k+i+1}}, \tag{5}$$

$$y_k = \sum_{i=0}^{k} \frac{(k+i-1)!}{i!(k-i)!k!} \frac{(-1)^{k+1} k \sin^{(k-i)}(\alpha) + (-1)^i i \sin^{(k-i)}(-\alpha)}{(-2\alpha)^{k+i+1}}. \tag{6}$$

The above simple expressions are obtained by exploiting the symmetry of the trigonometric functions and the parameter domain $t \in [-\alpha, \alpha]$. Note that the exact degree of $X_{2n+1}(\alpha; t)$ is $2n$ rather than $2n+1$. After some calculations, (5) and (6) can be further simplified to

$$x_k = \frac{1}{k!} \sum_{i=0}^{k-1} \frac{(k+i-1)!}{i!(k-i-1)!} \frac{\cos\left(\frac{k+i}{2}\pi + \alpha\right)}{(2\alpha)^{k+i+1}}, \tag{7}$$

$$y_k = \frac{1}{k!} \sum_{i=0}^{k} \frac{(k+i)!}{i!(k-i)!} \frac{\sin\left(\frac{k+i}{2}\pi + \alpha\right)}{(2\alpha)^{k+i+1}}. \tag{8}$$

We can easily derive the following relation between coefficients,

$$y_k = -2\alpha(k+1)x_{k+1}, \quad k = 1, 2, \ldots. \tag{9}$$

Note that

$$x_1 = -\frac{\sin\alpha}{4\alpha^2}, \qquad y_1 = \frac{\cos\alpha}{4\alpha^2} - \frac{\sin\alpha}{4\alpha^3}.$$

Thus, the cubic Hermite interpolants of the trigonometric functions are

$$X_3(\alpha; t) = \cos\alpha + \frac{\alpha\sin\alpha}{2} - \frac{\sin\alpha}{2\alpha} t^2$$

and

$$Y_3(\alpha; t) = \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha} t + \frac{\alpha\cos\alpha - \sin\alpha}{2\alpha^3} t^3.$$

Finally, for the circular arc $\mathbf{c}(\alpha; t)$ in (1), we can obtain the degree-$(2n+1)$ Hermite interpolating polynomial

$$\begin{aligned}\mathbf{C}_{2n+1}(\alpha; t) &= \left(rX_{2n+1}(\alpha; t), rY_{2n+1}(\alpha; t)\right)\\ &= \mathbf{C}_1(\alpha; t) + \left(2r\alpha \sum_{k=1}^{n} x_k \left(t^2-\alpha^2\right)^k, 2rt \sum_{k=1}^{n} y_k \left(t^2-\alpha^2\right)^k\right),\end{aligned} \tag{10}$$

where

$$\mathbf{C}_1(\alpha; t) = \left(r\cos\alpha, \frac{r\sin\alpha}{\alpha} t\right) \tag{11}$$

is the degree-1 Hermite interpolating polynomial. And for the helix segment $\mathbf{h}(\alpha; t)$ in (1), we can obtain the corresponding degree-$(2n+1)$ Hermite interpolating polynomial by

$$\mathbf{H}_{2n+1}(\alpha; t) = \left(rX_{2n+1}(\alpha; t), rY_{2n+1}(\alpha; t), pt\right). \tag{12}$$

To measure the approximation quality of $\mathbf{c}(\alpha; t)$ by $\mathbf{C}_{2n+1}(\alpha; t)$, we define the error function by

$$\varepsilon_{2n+1}(\alpha; t) = \|\mathbf{c}(\alpha; t) - \mathbf{C}_{2n+1}(\alpha; t)\|, \quad t \in [-\alpha, \alpha]. \tag{13}$$

In fact, for any $t \in [-\alpha, \alpha]$, we have $\|\mathbf{h}(\alpha; t) - \mathbf{H}_{2n+1}(\alpha; t)\| = \|\mathbf{c}(\alpha; t) - \mathbf{C}_{2n+1}(\alpha; t)\|$. So, the approximation error between $\mathbf{h}(\alpha; t)$ and $\mathbf{H}_{2n+1}(\alpha; t)$ equals to that between $\mathbf{c}(\alpha; t)$ and $\mathbf{C}_{2n+1}(\alpha; t)$. In the following theorem, we propose an upper error bound for the polynomial approximation of $\mathbf{c}(\alpha; t)$ by $\mathbf{C}_{2n+1}(\alpha; t)$.

Theorem 1. *$\|\mathbf{c}(\alpha; t) - \mathbf{C}_{2n+1}(\alpha; t)\|$ converges uniformly to zero on $[-\alpha, \alpha]$ and has an upper error bound*

$$\overline{\varepsilon}_{2n+1} = \frac{\sqrt{2}r}{(2n+2)!}\alpha^{2n+2}. \tag{14}$$

Proof. By Theorem 3.5.1 in [15, p. 67], we obtain

$$\mathbf{c}(\alpha; t) - \mathbf{C}_{2n+1}(\alpha; t) = \frac{\left(t^2-\alpha^2\right)^{n+1}}{(2n+2)!}\left(r\cos^{(2n+2)}(\xi_1), r\sin^{(2n+2)}(\xi_2)\right),$$

where $\xi_1, \xi_2 \in (-\alpha, \alpha)$. Then, we have

$$\|\mathbf{c}(\alpha; t) - \mathbf{C}_{2n+1}(\alpha; t)\| = \frac{r\left(\alpha^2-t^2\right)^{n+1}}{(2n+2)!}\sqrt{\cos^2\xi_1 + \sin^2\xi_2} \le \frac{\sqrt{2}r}{(2n+2)!}\alpha^{2n+2}.$$

This completes the proof. □

For any user-prescribed tolerance δ, it is easy to calculate the lowest n such that $\overline{\varepsilon}_{2n+1} \le \delta$. So we can obtain the polynomial curve of a lowest degree $2n+1$ that approximates the circular arc within the tolerance. It can be considered as an adaptive method, i.e., the user can choose a smaller δ for higher accuracy, and also can choose a larger δ for lower accuracy. Therefore, it brings a convenient interactive editing tool for geometric shape design. If the degree of the approximating polynomial is too high for a relatively small tolerance, we can subdivide the circular arc into several segments and then reduce the degree for each segment. The error bound (14) will decrease exponentially by a factor $1/4^{n+1}$ if α is replaced by $\alpha/2$. And the resulting piecewise polynomial curves are C^n continuous at joint points. Therefore, we can achieve a good balance between accuracy and efficiency.

3. Examples and discussions

In this section, we show two examples to demonstrate the effectiveness of our method.

In the first example, we consider the unit circle defined by $\mathbf{c}(\pi; t) = (\cos t, \sin t)$, $t \in [-\pi, \pi]$. As displayed in Fig. 1, the successive polynomial curves $\mathbf{C}_{2n+1}(\alpha; t)$ gradually approximate the circle. When $n = 5$, we obtain a good enough approximation, i.e., one cannot distinguish the approximating curve from the circle. However, the polynomial degree is 11 in such a case. If we subdivide the circle into four segments, a good enough approximation will be achieved when $n = 2$, see Fig. 2. The error functions are plotted on the right of Figs. 1 and 2. In Table 1, we list the maximum errors between $\mathbf{c}(\alpha; t)$ and $\mathbf{C}_{2n+1}(\alpha; t)$, which are obtained by sampling-based errors, i.e.,

$$\max\left\{\varepsilon_{2n+1}(\alpha; t_i) : t_i = -\alpha + \frac{2i}{N}\alpha,\ 0 \le i \le N\right\},$$

where the positive integer N denotes the sampling density. The values in the column $n = 1$ decrease by about $1/4^2$, those in the column $n = 2$ by about $1/4^3$, those in the column $n = 3$ by about $1/4^4$, and so on. Therefore, a polynomial approximation

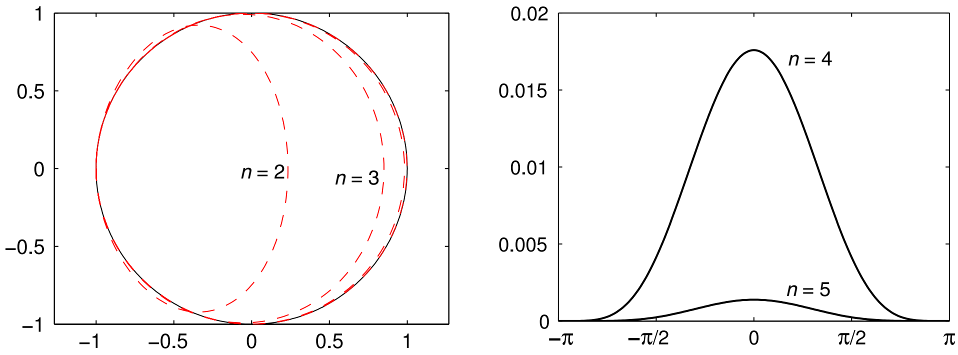

Fig. 1. Polynomial approximation of the unit circle. Left: the polynomial curves; Right: the error functions.

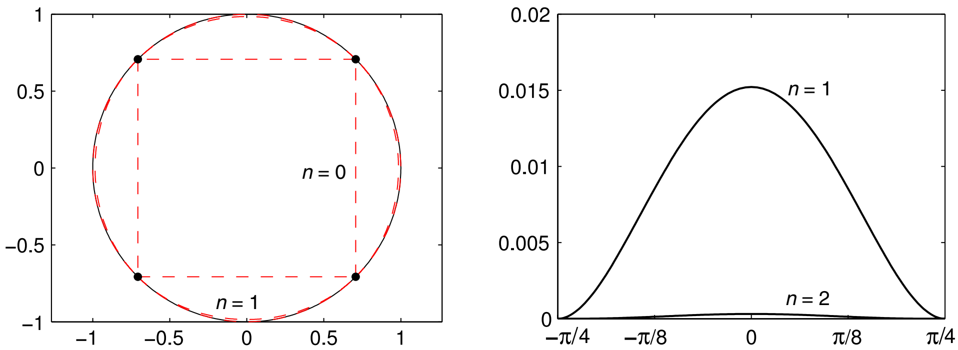

Fig. 2. Polynomial approximation of the unit circle by four segments. Left: the polynomial curves; Right: the error functions.

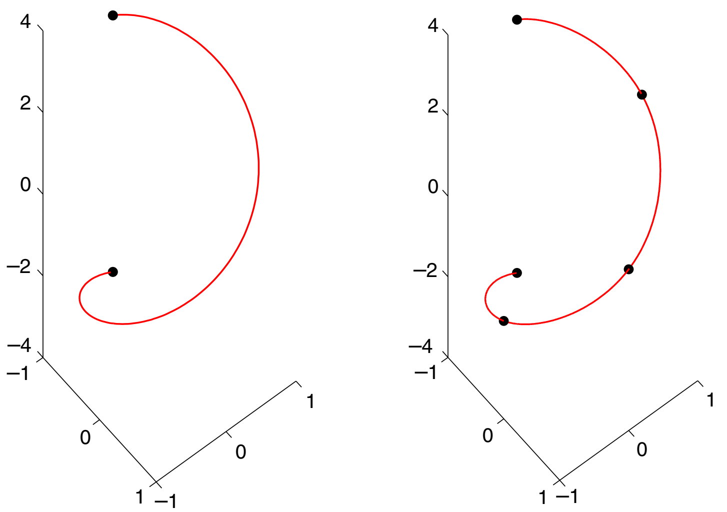

Fig. 3. Polynomial approximation of the helix segment. Left: $\mathbf{H}_9(\pi; t)$; Right: four segments based on $\mathbf{H}_3(\pi/4; t)$.

can always be achieved within any prescribed tolerance and with the constraint on the maximum degree of polynomials by subdividing the circle into several segments.

In the second example, we consider the helix segment $\mathbf{h}(\pi; t) = (\cos t, \sin t, t),\ t \in [-\pi, \pi]$. Note that the projection of $\mathbf{h}(\pi; t)$ onto the xy-plane is the unit circle. The results are displayed in Fig. 3. In addition, the maximum errors between $\mathbf{h}(\alpha; t)$ and $\mathbf{H}_{2n+1}(\alpha; t)$ agree with those listed in Table 1.

Table 1
The maximum errors between $\mathbf{c}(\alpha; t)$ and $\mathbf{C}_{2n+1}(\alpha; t)$.

α	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
π	2	7.6630e−01	1.4945e−01	1.7587e−02	1.3848e−03
$\pi/2$	2.1460e−01	1.8252e−02	8.2304e−04	2.2940e−05	4.3412e−07
$\pi/4$	1.5213e−02	3.1537e−04	3.4936e−06	2.4042e−08	1.1269e−10
$\pi/8$	9.8075e−04	5.0517e−06	1.3931e−08	2.3895e−11	2.7978e−14
$\pi/16$	6.1772e−05	7.9424e−08	5.4698e−11	2.3426e−14	6.8302e−18
$\pi/32$	3.8682e−06	1.2429e−09	2.1383e−13	2.2859e−17	1.6671e−21

4. Conclusions

In this paper, we have constructed a simple method for the polynomial approximation of circular arcs and helices by using the two-point Taylor expansion. Theoretical analysis and experiments show the effectiveness of the proposed method. Furthermore, the approximation error decreases rapidly and converges to zero as the polynomial degree tends to ∞. However, our method has an intrinsic limitation due to the degree-$(2n + 1)$ Hermite interpolant: for a specific n, our method cannot do better than those methods that make use of geometric data (tangents, curvatures, etc.), e.g. [2,11]. To compensate for it, our method has the good ability to reduce the approximation error by increasing the polynomial degree.

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